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A DEFINITION OF SENSE ON CLOSED CURVES IN NON-METRICAL PLANE ANALYSIS SITUS.*

By John Robert Kline.

1. Introduction.

One of the main aims of research in the foundations of geometry is to free geometry from any dependence upon the intuition. It is the purpose of the present paper to give a rigorous definition of a notion, which is very frequently left to the intuition—the notion of sameness of sense† on closed curves.‡ All the theorems of the present paper are proved on the basis of Professor R. L. Moore's system of axioms Σ_3.§ *Hence all the results hold even in certain spaces which are neither metrical, descriptive nor separable.*

The definition proposed is as follows: *The sense $A_1B_1C_1$ on the closed curve J_1 and the sense $A_2B_2C_2$ on the closed curve*|| *J_2 are said to be the same with respect to their common exterior E_{12}, if there exists in E_{12} a simple closed curve J_3 and three distinct points A_3, B_3 and C_3 thereon such that* (1) *if it is impossible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 by arcs, no two of which have a point in common and which lie except for their endpoints in E_{13} (the common exterior of J_1 and J_3), then it is also impossible to join*

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* Presented to the Society, April 29, 1916.

† I wish to thank Professor R. L. Moore, who suggested the undertaking of this investigation. The general definition given in this paper for sameness of sense on any two closed curves is a modification of the following definition suggested by him for the case where the closed curves have no point in common: If the distinct points A_i, B_i and C_i $(i = 1, 2)$ of the non-intersecting closed curves J_1 and J_2 are such that there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two of which have a point in common and which have, except for their endpoints, no point in common with either J_1 or J_2, then the sense $A_1B_1C_1$ on J_1 is said to be *opposite* to the sense $A_2B_2C_2$ on J_2 with respect to the region in which these arcs are drawn. If such a joining is not possible then the senses are said to be the *same.*

‡ If A and B are distinct points, then a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B but containing no proper connected subset containing both A and B. See N. J. Lennes, Curves in non-metrical analysis situs with an application in the calculus of variations, American Journal of Mathematics, vol. 33 (1911), p. 308. In the present paper "arc" and "simple continuous arc" will be considered synonymous terms. A *simple closed curve* is a set of points composed of two arcs AXB and AYB which have no point in common other than A and B.

§ Cf. R. L. Moore, On the foundations of plane analysis situs, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–64.

|| Hereafter in this paper, "closed curve" and "simple closed curve" will be used synonymously. In the present definition A_i, B_i and C_i $(i = 1, 2, 3)$ are distinct points of the closed curve J_i.

A_2 to A_3, B_2 to B_3 and C_2 to C_3 by arcs, no two of which have a point in common and which lie except for their endpoints in E_{23} (the common exterior of J_2 and J_3), (2) if it is possible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 as indicated above, then it is also possible to join A_2 to A_3, B_2 to B_3 and C_2 to C_3 as indicated. Otherwise the sense $A_1B_1C_1$ on J_1 and the sense $A_2B_2C_2$ on J_2 are said to be opposite with respect to E_{12}.

2. Certain Theorems in Analysis Situs.

Before establishing certain properties of sense, we shall first prove a number of preliminary theorems in analysis situs.

Theorem A: *Suppose the arc $A_1X_2A_2$ is such that $\underparen{A_1X_2A_2}$* has no point in common with the arcs $A_1B_1C_1$ and $A_2B_2C_2$ of the closed curve $A_1B_1C_1$-$ZC_2B_2A_2XA_1$. Then there are on the arc $A_1X_2A_2$ at most a finite number of distinct subarcs $K_iP_iL_i$ such that* (1) *K_i is a point of A_1XA_2 while L_i is a point of C_1ZC_2,* (2) *no point of $\underparen{K_iP_iL_i}$ is on $A_1B_1C_1ZC_2B_2A_2XA_1$.*

Proof: Suppose Theorem A is false. Then there exists at least one arc $A_1X_1A_2$ such that (1) $\underparen{A_1X_1A_2}$ has no point in common with either the arc $A_1B_1C_1$ or the arc $A_2B_2C_2$ of $A_1B_1C_1ZC_2B_2A_2XA_1$, (2) there are on $A_1X_1A_2$ infinitely many distinct points $[K]$ such that (a) to each point K' of $[K]$ there corresponds a point $L_{k'}$, common to $A_1X_1A_2$ and C_1ZC_2 and an arc $K'P_{k'}L_{k'}$, which is a subset of $A_1X_1A_2$, (b) each point of $[K]$ is on A_1XA_2, (c) $\underparen{K'P_{k'}L_{k'}}$ has no point in common with the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$, (d) the order $A_1K'L_{k'}A_2$ holds on $A_1X_1A_2$. As the point set $[K]$ is a subset of $A_1X_1A_2$, it is bounded. Hence $[K]$ has at least one limit point $\bar{K}$.† Hence there exists a sequence of points K_1, K_2, K_3, $\cdots$, such that (1) each point of the sequence belongs to $[K]$, (2) $\bar{K}$ is the only limit point of K_1, K_2, K_3, $\cdots$,‡ (3) either the order $A_1K_1K_2K_3\cdots\bar{K}A_2$ or the order $A_1\bar{K}\cdots K_3K_2K_1A_2$ holds on A_1XA_2. To every point K_i there corresponds a point L_i, common to $A_1X_1A_2$ and C_1ZC_2 and an arc $K_iP_iL_i$, such that $\underparen{K_iP_iL_i}$ is a subset of $A_1X_1A_2$ and has no point in common with $A_1B_1C_1ZC_2B_2A_2XA_1$. For all values of i the order $A_1K_iL_iA_2$ holds on $A_1X_1A_2$. As the arc $K_iP_iL_i$ and the arc

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* If AXB is an arc, then $\underparen{AXB}$ denotes the point set $AXB - A - B$. Likewise if AB is an arc, then $\underparen{AB}$ denotes the point set $AB - A - B$.

† Cf. F. Hausdorff, Grundzüge der Mengenlehre, Veit and Co., Leipzig, 1914, p. 231. See also E. W. Chittenden, The Heine Borel Theorem in a Riesz Domain, Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 179–83, and vol. 20 (1914), p. 461. For a proof that, in the presence of certain linear order axioms, the Heine Borel Theorem is equivalent to the Dedekind Cut Postulate, see O. Veblen, The Heine-Borel Theorem, Bulletin of the American Mathematical Society, vol. 10 (1903–4), pp. 463–9.

‡ Cf. R. L. Moore, loc. cit., Theorems 8 and 9, p. 134.

$K_jP_jL_j$ $(i \neq j)$ have no point in common, either the order $A_1K_1L_1K_2L_2 \cdots \bar{K}A_2$ or the order $A_1\bar{K} \cdots K_2L_2K_1L_1A_2$ holds on $A_1X_1A_2$.

Case I. Suppose $A_1K_1L_1K_2L_2 \cdots \bar{K}A_2$ holds. It may easily be shown that $\bar{K}$ is a limit point of the set $L_1, L_2, L_3, \cdots$. As every point of $L_1, L_2, L_3, \cdots$, belongs to C_1ZC_2, a closed point set, $\bar{K}$ belongs to C_1ZC_2. But $\bar{K}$ is on A_1XA_2. Hence C_1ZC_2 and A_1XA_2 have at least one point in common, contrary to hypothesis. Thus, in Case I, we are led to a contradiction.

Case II. Suppose $A_1\bar{K} \cdots K_2L_2K_1L_1A_2$ holds on $A_1X_1A_2$. This is impossible as in Case I.

It may easily be seen that if $A_1X_1A_2$ has any point in common with C_1ZC_2, then there exist $2n$ distinct subarcs of $A_1X_1A_2$, $K_1P_1L_1$, $L_2P_2K_2$, $\cdots$, $L_{2n}P_{2n}K_{2n}$, where $K_1, K_2, \cdots, K_{2n}$ is a subset of A_1XA_2 while $L_1, L_2, \cdots, L_{2n}$ is a subset of C_1ZC_2 and the set $\underparen{K_iP_iL_i}$ $(i = 1, 2, \cdots, 2n)$ has no point in common with $A_1B_1C_1ZC_2B_2A_2XA_1$. On $A_1X_1A_2$ the order $K_1L_1K_2L_3K_4 \cdots L_{2n-1}K_{2n}$ holds. If $K_1 \neq A_1$, then there is on the subarc A_1K_1 of $A_1X_1A_2$ no point of C_1ZC_2. Likewise if $K_{2n} \neq A_2$, there is on the subarc $K_{2n}A_2$ of $A_1X_1A_2$ no point of C_1ZC_2. If $K_{2i+1} \neq K_{2i}$ $(i = 1, 2, \cdots, n - 1)$ then K_{2i+1} is a point of the subset $\underparen{K_{2i}L_{2i+1}}$ of $A_1X_1A_2$. If $L_{2i} \neq L_{2i-1}$ $(i = 1, 2, \cdots, n)$, then L_{2i} is a point of the subset $\underparen{L_{2i-1}K_{2i}}$ of $A_1X_1A_2$.

Theorem B: *Suppose the closed curves* J_1 *and* J_2 *are such that* $J_1 + I_1$* *is a subset of* E_2, (2) A_i, B_i *and* C_i *are distinct points of* J_i, (3) A_1XA_2 *and* C_1ZC_2 *are arcs which have no point in common and lie except for their endpoints in* E_{12}. *Under these conditions, in order that there exist an arc* B_1YB_2, *which fails to meet either* A_1XA_2 *or* C_1ZC_2 *and lies except for its endpoints in* E_{12}, *it is both necessary and sufficient that* I_1 *and* I_2 *be either both within or both without* $A_1B_1C_1ZC_2B_2A_2XA_1$.

Proof: *The condition is sufficient.* Suppose I_1 and I_2 are both within $A_1B_1C_1ZC_2B_2A_2XA_1$.† Let D_i $(i = 1, 2)$ denote a point of J_i such that A_i and C_i separate B_i and D_i on J_i. All points of $\underparen{A_iD_iC_i}$ are within $A_1B_1C_1ZC_2B_2A_2XA_1$. For suppose some point P of $\underparen{A_1D_1C_1}$ were without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exists a region R containing P and lying entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$. Hence R contains no point of I_1, which is entirely within $A_1B_1C_1ZC_2B_2A_2XA_1$. But every region containing a point of J_i must contain a point of I_i. Hence the supposition that a point of $\underset{\smile}{A_1D_1C_1}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$ leads to a contradiction. In like manner all points of $\underset{\smile}{A_2D_2C_2}$ are within $A_1B_1C_1ZC_2B_2A_2XA_1$. Let M denote any point without this closed curve. Join M to B_1 and M to B_2 by arcs MB_1 and MB_2, lying except for B_1 and B_2, respectively, entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$.[*] The point set $MB_1 + MB_2$ will contain as a subset an arc B_1YB_2, which has no point in common with $A_1XA_2 + C_1ZC_2$ and lies, except for its endpoints, entirely in E_{12}. In case I_1 and I_2 are both without $A_1B_1C_1ZC_2B_2A_2XA_1$, it follows that $\underset{\smile}{A_1D_1C_1} + \underset{\smile}{A_2D_2C_2}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exists an arc B_1YB_2 such that $\underset{\smile}{B_1YB_2}$ is a subset of the interior of $A_1B_1C_1ZC_2B_2A_2XA_1$.

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* If J_i is a closed curve, I_i will denote the interior of J_i while E_i denotes its exterior. The symbol E_{ij} will denote the common exterior of J_i and J_j.

† That I_i $(i = 1, 2)$ lies either entirely within or entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$ follows at once from the following facts: (1) a region is a connected point set; (2) no point $\left\{\begin{matrix}\text{within}\\\text{without}\end{matrix}\right\}$ a region is a limit point of a set of points lying entirely $\left\{\begin{matrix}\text{without}\\\text{within}\end{matrix}\right\}$ that region.

(*b*) *The condition is necessary.* Suppose there is an arc B_1YB_2, which fails to meet either A_1XA_2 or C_1ZC_2 and such that $\underset{\smile}{B_1YB_2}$ lies in E_{12}, while I_1 is within and I_2 is without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then it may easily be proved that $\underset{\smile}{A_1D_1C_1}$ is within and $\underset{\smile}{A_2D_2C_2}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$ while $\underset{\smile}{B_1YB_2}$ is either entirely within or entirely without this closed curve. But if $\underset{\smile}{B_1YB_2}$ is entirely within $A_1B_1C_1ZC_2B_2A_2XA_1$, then it has at least one point in common with $\underset{\smile}{A_1D_1C_1}$ while if it lies entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$, it has at least one point in common with $\underset{\smile}{A_2D_2C_2}$.[†] But this is contrary to our assumption. In like manner, if we suppose I_1 is without and I_2 is within $A_1B_1C_1ZC_2B_2A_2XA_1$, we are led to a contradiction.

Theorem C: *Suppose the closed curve J_1 and its interior I_1 are without the closed curve J_2,* (2) *A_i and C_i separate B_i and D_i on $J_i (i=1, 2)$,* (3) *there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two of which have a point in common and which lie except for their endpoints in E_{12}. Under these conditions, there exists an arc D_1WD_2, which fails to meet $A_1XA_2 + C_1ZC_2$ and lies except for its endpoints in E_{12}.*

Theorem D: *If the points A and C separate B and D on the closed curve J while K is a closed point set such that K has no point in common with $\underset{\smile}{ABC}$, then there exists an arc AFC such that $\underset{\smile}{AFC}$ is within J and fails to meet K.*

Proof: Put about B a closed curve $\overline{C}$ such that neither $\overline{C}$ nor its interior contains a point of $ADC + K$. Let C_1 be the first point of the subarc BC of ABC which is on $\overline{C}$ while A_1 is the first point of the subarc BA of

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[*] Cf. R. L. Moore, loc. cit., Theorem 39, pp. 153–5.

[†] Cf. R. L. Moore, loc. cit., Theorem 29, pp. 145–6.

CBA on $\bar{C}$. There exists a closed curve J_1 such that (1) B is on J_1, (2) all points on J_1 are points either of J or of $\bar{C}$, (3) every point within J_1 is within J and within $\bar{C}$.[*] All points of A_1BC_1 are on J_1. There exists an arc $A_1X_1C_1$ such that $\underset{\smile}{A_1X_1C_1}$ is within J_1 and hence within J and fails to meet either ADC or K. Let Q be a point within $A_1BC_1X_1A_1$.

If in the proof of Theorem 32[†] of Prof. Moore's paper we insist that for all values of i greater than 1, the interior of $B_iX_iC_iC_{i-1}B_i$[‡] fails to contain any point of $K + Q + C$, while (1) $B_iX_iC_i$ is within J and fails to meet K, (2) $C_iC_{i-1}B_i$ is a subset of the arc BC of ABC, we obtain a proof that there exists an arc QC such that $\underset{\smile}{QC} + Q$ is a subset of the interior of J and fails to meet K. In the same manner we may prove that there exists an arc QA such that $\underset{\smile}{QA} + Q$ is a subset of the interior of J and fails to meet K. The point set $QC + QA$ contains as a subset an arc AQ_1C, such that $\underset{\smile}{AQ_1C}$ is within J and fails to meet K.

A similar proof will show that there exists an arc AGC such that $\underset{\smile}{AGC}$ is without J and fails to meet K.

Theorem E: *Suppose* (1) *the closed curve* J_1 *and its interior* I_1 *are without the closed curve* J_2, (2) A_i *and* C_i ($i = 1, 2$) *separate* B_i *and* D_i *on* J_i, (3) $A_1X_1A_2$, $A_1X_2A_2$ *and* $C_1Z_1C_2$ *are arcs lying except for their endpoints in* E_{12} *and such that neither* $A_1X_1A_2$ *nor* $A_1X_2A_2$ *meets* $C_1Z_1C_2$, (4) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. *Then* I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

Proof: Case I. I_1 and I_2 are both without $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then it follows that $\underset{\smile}{A_1D_1C_1}$ and $\underset{\smile}{A_2D_2C_2}$ are both without $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. There are three possibilities:

(*a*) $\underset{\smile}{A_1X_2A_2}$ is a subset of the interior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then the interior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1 = \underset{\smile}{A_1X_2A_2} +$ the interior of $A_1X_1A_2X_2A_1 +$ the interior of $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$. Hence I_1 and I_2 are both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

(*b*) $\underset{\smile}{A_1X_2A_2}$ is a subset of the exterior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. By Theorem B, there exists an arc B_1YB_2 such that $\underset{\smile}{B_1YB_2}$ is a subset of the interior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then, as $\underset{\smile}{B_1YB_2}$ has no point in common with either $A_1X_2A_2$ or $C_1Z_1C_2$ while $\underset{\smile}{B_1YB_2}$ lies in E_{12}, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

(*c*) $\underset{\smile}{A_1X_2A_2}$ is not a subset either of the interior or of the exterior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then, by Theorem D, there exists an arc A_1EA_2

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[*] Cf. R. L. Moore, loc. cit., Theorem 43, pp. 156–7.

[†] Cf. R. L. Moore, loc. cit., Theorem 32, pp. 147–8.

[‡] B_i corresponds to A_i and C_i to B_i in Professor Moore's notation.

such that $\underparen{A_1EA_2}$ is within $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$ and fails to meet $A_1X_2A_2$. By (*a*), I_1 and I_2 are both without $A_1B_1C_1Z_1C_2B_2A_2EA_1$. By (*b*) I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

Case II. I_1 and I_2 are both within $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then I_1 and I_2 are both without $A_1D_1C_1Z_1C_2D_2A_2X_1A_1$. By Case I, I_1 and I_2 are either both within or both without $A_1D_1C_1Z_1C_2D_2A_2X_2A_1$. Then, by Theorems B and C, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

Theorem F: *Suppose* (1) *the closed curve* J_1 *and its interior* I_1 *are without the closed curve* J_2, (2) A_i *and* C_i ($i = 1, 2$) *separate* B_i *and* D_i *on* J_i, (3) $A_1X_1A_2$, $A_1X_2A_2$ *and* $C_1Z_1C_2$ *are arcs lying except for their endpoints in* E_{12}, *while* $A_1X_1A_2$ *and* $C_1Z_1C_2$ *have no points in common,* (4) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. *Under these conditions there exist arcs* A_1XA_2 *and* C_1ZC_2 *such that* (1) A_1XA_2 *and* C_1ZC_2 *have no points in common and lie except for their endpoints in* E_{12}, (2) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1ZC_2B_2A_2XA_1$, (3) *any point of* $A_1X_2A_2$, *which is not on* $A_1B_1C_1ZC_2B_2A_2XA_1$, *is a point of one of* $2n$ *distinct subarcs of* $A_1X_2A_2$, $K_iP_iL_i$ ($i = 1, 2, \cdots, 2n$) *where* K_i *is a point common to* $A_1X_2A_2$ *and* A_1XA_2, L_i *is a point common to* $A_1X_2A_2$ *and* C_1ZC_2, *while no point of* $\underparen{K_iP_iL_i}$ *is on* $A_1B_1C_1ZC_2B_2A_2XA_1$, (4) *on* $A_1X_2A_2$, *the order* $K_1L_1K_2L_3\cdots K_{2n}$ *holds, while if* $K_{2j+1} \neq K_{2j}$ ($j = 1, 2, \cdots, n-1$) *then* K_{2j+1} *is a point of the subset* $\underparen{K_{2j}L_{2j+1}}$ *on* $A_1X_2A_2$ *and if* $L_{2m} \neq L_{2m-1}$ ($m = 1, 2, \cdots, n$) *then* L_{2m} *is a point of the subset* $\underparen{L_{2m-1}K_{2m}}$ *on* $A_1X_2A_2$, (5) *if* $K_1 \neq A_1$ *then the arc* A_1K_1 *of* A_1XA_2 *is the arc* K_1A_1 *of* $A_1X_2A_2$ *and if* $K_{2n} \neq A_2$ *then the arc* $K_{2n}A_2$ *on* A_1XA_2 *is the arc* $K_{2n}A_2$ *of* $A_1X_2A_2$, (6) *on* A_1XA_2 *the order* $K_1K_2K_4\cdots K_{2n}$ *holds while if* $K_{2j+1} \neq K_{2j}$, *then the arc* $K_{2j}K_{2j+1}$ *of* A_1XA_2 *is the arc* $K_{2j}K_{2j+1}$ *of* $A_1X_2A_2$ *while* K_{2j+1} *is a point of* $\underparen{K_{2j}K_{2(j+1)}}$ *on* A_1XA_2, (7) *on* C_1ZC_2 *the order* $C_1L_1L_3\cdots L_{2n-1}C_2$ *holds while if* $L_{2m} \neq L_{2m-1}$, *then* L_{2m} *is a point of the subset* $\underparen{L_{2m-1}L_{2m+1}}$ *of* C_1ZC_2* *and the arc* $L_{2m-1}L_{2m}$ *of* C_1ZC_2 *is the arc* $L_{2m-1}L_{2m}$ *of* $A_1X_2A_2$.

Proof: Let L_1 denote the first point which $A_1X_2A_2$ has in common with $C_1Z_1C_2$. [If no such point exists, then $A_1X_2A_2$ fails to meet $C_1Z_1C_2$. Then by Theorem E, $A_1X_2A_2$ may be taken as A_1XA_2 while $C_1Z_1C_2$ is C_1ZC_2.] Let K_1 denote the last point which the arc $A_1X_1A_2$ has in common with the arc A_1L_1 of $A_1X_2A_2$ (Fig. 1). Let $A_1X_{11}A_2$ denote the arc composed of A_1K_1 (on $A_1X_2A_2$, if $K_1 \neq A_1$) $+$ K_1A_2 (on $A_1X_1A_2$). Then, as $A_1X_{11}A_2$ fails to meet $C_1Z_1C_2$ and I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$, it follows by Theorem E, that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_{11}A_1$. Let P_1 denote a point between K_1 and L_1 on $A_1X_2A_2$.

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* If $m = n$ and $L_{2n-1} \neq L_{2n}$, then L_{2n} is on $\underparen{L_{2n-1}C_2}$ of C_1ZC_2.

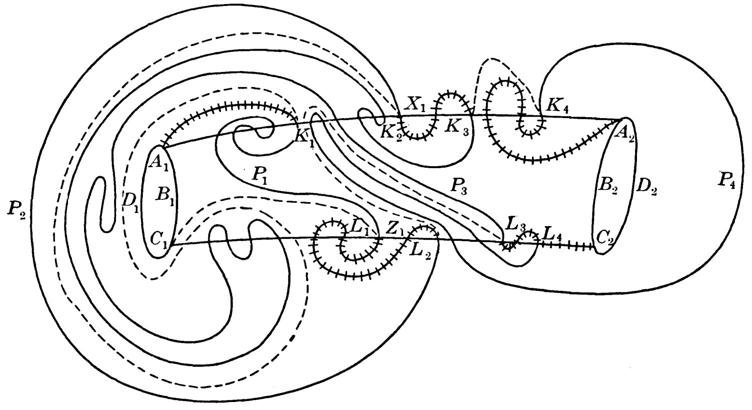

FIG. 1.

If A_2 is without $A_1B_1C_1L_1P_1K_1A_1$, then all points of $\underparen{K_1A_2}$ (on $A_1X_{11}A_2$) $+ J_2 + \underparen{C_2L_1}$ (on $C_1Z_1C_2$) are without $A_1B_1C_1L_1P_1K_1A_1$. As the arc L_1A_2 (on $A_1X_2A_2$) fails to have any point in common with $\underbrace{C_1B_1A_1K_1P_1L_1}$, then, by Theorem D, there exists an are $C_1M_1L_1$ such that $\underparen{C_1M_1L_1}$ is within $A_1B_1C_1L_1P_1K_1A_1$ and fails to meet either $\underparen{L_1A_2}$ (on $A_1X_2A_2$) or $\underparen{A_1D_1C_1}$. If A_2 is within $A_1B_1C_1L_1P_1K_1A_1$, then all points of $\underparen{K_1A_2}$ (on $A_1X_{11}A_2$) $+ J_2 + \underparen{C_2L_1}$ (on $C_1Z_1C_2$) are within $A_1B_1C_1L_1P_1K_1A_1$. In this case let $C_1M_1L_1$ denote an arc such that $\underparen{C_1M_1L_1}$ is without $A_1B_1C_1$-$L_1P_1K_1A_1$ and fails to meet either $\underparen{L_1A_2}$ or $\underparen{A_1D_1C_1}$. Let $C_1Z_{11}C_2$ denote the arc composed of $C_1M_1L_1 + L_1C_2$ (on $C_1Z_1C_2$). As $C_1Z_{11}C_2$ fails to meet $A_1X_{11}A_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_{11}A_1$, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{11}C_2B_2A_2X_{11}A_1$. Let K_2 denote the first point of the arc L_1A_2 of $A_1X_2A_2$ which is on $A_1X_{11}A_2$. Let L_2 denote the last point of the arc $C_1Z_{11}C_2$ which is on the arc L_1K_2 of $A_1X_2A_2$. Clearly L_2 either is L_1 or is a point of $\underparen{L_1C_2}$ on $C_1X_{11}C_2$. Let $C_1Z_{12}C_2$ denote the arc composed of C_1L_1 (on $C_1Z_{11}C_2$) $+ L_1L_2$ (on $A_1X_2A_2$, if $L_1 \neq L_2$) $+ L_2C_2$ (on $C_1Z_{11}C_2$). As $C_1Z_{12}C_2$ fails to meet $A_1X_{11}A_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{11}C_2B_2A_2X_{11}A_1$, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{12}C_2B_2A_2$-$X_{11}A_1$. Let P_2 denote a point between L_2 and K_2 on $A_1X_2A_2$.

If A_2 is without $K_1P_1L_1L_2P_2K_2K_1$, then all points of $\underparen{K_2A_2}$ (on $A_1X_{11}A_2$) $+ J_2 + \underparen{C_2L_2}$ (on $C_1Z_{12}C_2$) are without $K_1P_1L_1L_2P_2K_2K_1$. Let $K_1F_2K_2$ denote an arc such that $\underparen{K_1F_2K_2}$ is within $K_1P_1L_1L_2P_2K_2K_1$ and fails to meet $\underparen{K_2A_2}$ (on $A_1X_2A_2$) $+ K_1A_1B_1C_1L_1 + A_1D_1C_1$. If A_2 is within $K_1P_1L_1L_2P_2K_2K_1$, then all points of $\underparen{K_2A_2}$ (on $A_1X_{11}A_2$) $+ J_2 + \underparen{C_2L_2}$ (on $C_1Z_{12}C_2$) are within $K_1P_1L_1L_2P_2K_2K_1$. In this case let $K_1F_2K_2$ denote an arc such that $\underparen{K_1F_2K_2}$ is without $K_1P_1L_1L_2P_2K_2K_1$, and fails to meet $\underparen{K_2A_2}$ (on $A_1X_2A_2$) $+ K_1A_1B_1C_1L_1 + A_1D_1C_1$. Let $A_1X_{12}A_2$ denote the arc composed of AK_1 (on $A_1X_{11}A_2$) $+ K_1F_2K_2 + K_2A_2$ (on $A_1X_{11}A_2$). As $A_1X_{12}A_2$ fails to meet $C_1Z_{12}C_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{12}C_2B_2A_2X_{11}A_1$, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{12}C_2B_2A_2X_{12}A_1$.

Continue this process, always being sure that if A_2 is $\left\{\begin{smallmatrix}\text{without}\\ \text{within}\end{smallmatrix}\right\}$ $L_{2i}P_{2i}K_{2i}K_{2i+1}P_{2i+1}L_{2i+1}L_{2i}$, then $L_{2i}M_{2i+1}L_{2i+1}$ is an arc such that $\underbrace{L_{2i}M_{2i+1}L_{2i+1}}$ is $\left\{\begin{smallmatrix}\text{within}\\ \text{without}\end{smallmatrix}\right\}$ $L_{2i}P_{2i}K_{2i}K_{2i+1}P_{2i+1}L_{2i+1}L_{2i}$ and fails to meet $\underparen{L_{2i+1}A_2}$ (on $A_1X_2A_2$) $+ L_{2i}A_1B_1C_1K_{2i} + A_1D_1C_1 + K_1P_1L_1 + K_2P_2L_2 + \cdots + K_{2i-1}P_{2i-1}L_{2i-1}$; and if A_2 is $\left\{\begin{smallmatrix}\text{without}\\ \text{within}\end{smallmatrix}\right\}$ $K_{2i+1}P_{2i+1}L_{2i+1}L_{2(i+1)}P_{2(i+1)}K_{2(i+1)}K_{2i+1}$, then $K_{2i+1}F_{2(i+1)}K_{2(i+1)}$ is an arc such that $\underbrace{K_{2i+1}F_{2(i+1)}K_{2(i+1)}}$ is $\left\{\begin{smallmatrix}\text{within}\\ \text{without}\end{smallmatrix}\right\}$ $K_{2i+1}P_{2i+1}L_{2i+1}L_{2(i+1)}P_{2(i+1)}K_{2(i+1)}K_{2i+1}$ and fails to meet $\underbrace{K_{2(i+1)}A_2}$ (on $A_1X_2A_2$) $+ K_{2i+1}A_1B_1C_1L_{2i+1} + A_1D_1C_1 + K_1P_1L_1 + K_2P_2L_2 + \cdots + K_{2i}P_{2i}L_{2i}$.

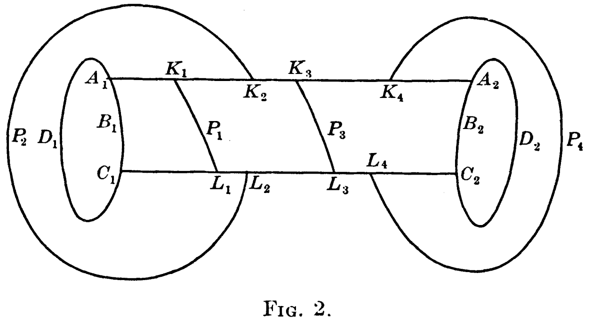

FIG. 2.

By Theorem A, there exists a certain finite number n such that on the arc $K_{2n}A_2$ of $A_1X_2A_2$, there is no point of the arc $C_1Z_{1,\,2n}C_2$. It is clear the K_{2n} is on the arc $K_{2n-1}A_2$ of $A_1X_{1,\,2n}A_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{1,\,2n}C_2B_2A_2X_{1,\,2n}A_2$. Let A_1XA_2 denote the arc composed of A_1K_{2n} (on $A_1X_{1,\,2n}A_2$) $+ K_{2n}A_2$ (on $A_1X_2A_2$) while C_1ZC_2 denotes $C_1Z_{1,\,2n}C_2$. As A_1XA_2 fails to meet C_1ZC_2 and I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{1,\,2n}C_2B_2A_2X_{1,\,2n}A_1$, it is clear that I_1 and I_2 are either both within or both without $A_1B_1C_1ZC_2B_2A_2XA_1$. It is clear that the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$ has the properties of the closed curve described in the conclusion of our theorem (Fig. 2).

Theorem G: *Under the same hypothesis as in Theorem F, there exists at least one arc $C_1Z_2C_2$ such that* (1) *$C_1Z_2C_2$ fails to meet $A_1X_2A_2$,* (2) *$\underset{\smile}{C_1Z_2C_2}$ is a subset of E_{12},* (3) *I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.*

Proof: Suppose the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$, which was described in Theorem F, has been constructed. We shall first prove that our theorem is true when $n = 1$.

Let us suppose I_1 and I_2 are both without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then $\underset{\smile}{A_1D_1C_1}$ and $\underset{\smile}{A_2D_2C_2}$ are both without $A_1B_1C_1ZC_2B_2A_2XA_1$ and the interior of $A_1D_1C_1ZC_2D_2A_2XA_1 = \underset{\smile}{A_1B_1C_1} + \underset{\smile}{A_2B_2C_2} + I_1 + I_2 +$ the interior of $A_1B_1C_1ZC_2B_2A_2XA_1$. There are a number of possible cases.

Case I. $\underset{\smile}{K_1P_1L_1}$ and $\underset{\smile}{K_2P_2L_2}$ are both within $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exist an arc $C_1Z_2C_2$ such that $\underset{\smile}{C_1Z_2C_2}$ is without $A_1D_1C_1ZC_2D_2A_2XA_1$. By Theorem E, I_1 and I_2 are both within or both without $A_1B_1C_1Z_2C_2B_2A_2XA_1$. But as $A_1X_2A_2$ fails to meet $C_1Z_2C_2$, it follows, by Theorem E, that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Case II. $\underset{\smile}{K_1P_1L_1}$ and $\underset{\smile}{K_2P_2L_2}$ are both without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exists an arc $C_1Z_2C_2$ such that $\underset{\smile}{C_1Z_2C_2}$ is within $A_1B_1C_1ZC_2B_2A_2XA_1$. By Theorem E, I_1 and I_2 are both within or both without $A_1B_1C_1Z_2C_2B_2A_2XA_1$. But as $A_1X_2A_2$ fails to meet $C_1Z_2C_2$, it follows, by Theorem E, that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Case III. $\underset{\smile}{K_1P_1L_1}$ is within and $\underset{\smile}{K_2P_2L_2}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then the interior of $A_1B_1C_1ZC_2B_2A_2XA_1 = \underset{\smile}{K_1P_1L_1} +$ the interior of $A_1B_1C_1L_1P_1K_1A_1 +$ the interior of $A_2B_2C_2L_1P_1K_1A_2$. As $\underset{\smile}{K_2P_2L_2}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$ and fails to contain any point of $\underset{\smile}{A_1D_1C_1} + \underset{\smile}{A_2D_2C_2} + I_1 + I_2$, it follows that $\underset{\smile}{K_2P_2L_2}$ is without $A_1D_1C_1ZC_2D_2A_2XA_1$. There are two possibilities:

(*a*) $\underset{\smile}{A_1D_1C_1}$ is without $A_2D_2C_2L_2P_2K_2A_2$. Then the interior of $A_1D_1C_1L_2P_2K_2A_1 = \underbrace{K_2A_2D_2C_2L_2} +$ the interior of $A_1D_1C_1ZC_2D_2A_2XA_1 +$ the interior of $A_2D_2C_2L_2P_2K_2A_2$.* Let M denote a point between K_1

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* Cf. R. L. Moore, Theorem 27, pp. 144–5.

and K_2 on A_1XA_2. Let C_1M be an arc such that $\underparen{C_1M}$ is a subset of the exterior of $A_1D_1C_1L_2P_2K_2A_1$ while MC_2 is an arc such that $\underparen{MC_2}$ is a subset of the interior of $A_2B_2C_2L_1P_1K_1A_2$. Let F denote a point between C_1 and L_1 on C_1ZC_2 while R denotes a point between M and K_2 on A_1XA_2. Let B_1F denote an arc such that $\underparen{B_1F}$ is a subset of the interior of $A_1B_1C_1L_1P_1K_1A_1$. As $\underparen{MC_2}$ is within $A_2B_2C_2L_1P_1K_1A_2$, the interior of $A_2B_2C_2L_1P_1K_1A_2 = \underparen{MC_2} +$ the interior of $MK_2A_2B_2C_2M +$ the interior of $K_1P_1L_1C_2MK_1$. Let RB_2 denote an arc such that $\underparen{RB_2}$ is a subset of the interior of $MK_2A_2B_2C_2M$. Let FR denote an arc such that $\underparen{FR}$ is within $C_1L_2P_2K_2MC_1$, provided $\underparen{A_1D_1C_1}$ is without $C_1L_2P_2K_2MC_1$, while $\underparen{FR}$ is without $C_1MK_2P_2L_2C_1$ provided $\underparen{A_1D_1C_1}$ is within $C_1MK_2P_2L_2C_1$. Let B_1YB_2 denote the arc composed of $B_1F + FR + RB_2$. Then B_1YB_2, $A_1X_2A_2$ and C_1MC_2 are arcs no two of which have any point in common and lying except for their endpoints in E_{12}. By Theorem B, I_1 and I_2 are either both within or both without $A_1B_1C_1MC_2B_2A_2X_2A_1$.

(*b*) $\underparen{A_1D_1C_1}$ is within $A_2D_2C_2L_2P_2K_2A_2$. Then the interior of $A_2D_2C_2L_2P_2K_2A_2 = \underbracket{K_2A_1D_1C_1L_2} +$ the interior of $A_1D_1C_1ZC_2D_2A_2XA_1 +$ the interior of $K_2A_1D_1C_1L_2P_2K_2$. Let M denote a point between K_1 and K_2 on A_1XA_2. Let C_1M be an arc such that $\underparen{C_1M}$ is within $A_1D_1C_1L_2P_2K_2A_1$, whose interior then is $\underparen{C_1M} +$ the interior of $C_1L_2P_2K_2MC_1 +$ the interior of $A_1D_1C_1MA_1$. Let C_2M denote an arc such that $\underparen{C_2M}$ is within $A_2B_2C_2L_1P_1K_1A_2$, whose interior then is $\underparen{MC_2} +$ the interior of $A_2B_2C_2MK_2A_2 +$ the interior of $K_1MC_2L_1P_1K_1$. Let F denote a point between C_1 and L_1 on C_1ZC_2 while B_1F is an arc such that $\underparen{B_1F}$ is within $A_1B_1C_1L_1P_1K_1A_1$. Let R denote a point between M and K_2 on A_1XA_2 while RF is an arc such that $\underparen{RF}$ is within $C_1L_2P_2K_2MC_1$. Let RB be an arc such that $\underparen{RB_2}$ is within $A_2B_2C_2MK_2A_2$. Let B_1YB_2 denote the arc $B_1F + FR + RB_2$. Then as $A_1X_2A_2$, B_1YB_2 and C_1MC_2 are arcs lying except for their endpoints in E_{12} and such that no two have any point in common, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1MC_2B_2A_2XA_1$.

Case IV. $\underparen{K_1P_1L_1}$ is without and $\underparen{K_2P_2L_2}$ is within $A_1B_1C_1Z_1C_2B_2A_2XA_1$. Case IV reduces to Case III if we read from A_2 to A_1 instead of from A_1 to A_2 on $A_1X_2A_2$.

It may readily be seen that in case I_1 and I_2 are both within $A_1B_1C_1ZC_2B_2A_2XA_1$, then I_1 and I_2 are both without $A_1D_1C_1ZC_2D_2A_2XA_1$. The argument just given shows that there exists an arc $C_1Z_2C_2$ such that (1) $\underparen{C_1Z_2C_2}$ fails to meet $A_1X_2A_2$ and is a subset of E_{12}, (2) I_1 and I_2 are either both within or both without $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. Hence by Theorems B and C, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Having now proved that our theorem is true when $n = 1$, let us assume that it holds true when $n = k - 1$, to prove it holds true for $n = k$. The arc $A_1\bar{\bar{X}}A_2$ composed of A_1K_2 (on A_1XA_2) + K_2A_2 (on $A_1X_2A_2$) will be an arc for which $n = k - 1$. Hence there exists an arc $C_1\bar{\bar{Z}}C_2$ such that (1) $C_1\bar{\bar{Z}}C_2$ fails to meet $A_1\bar{\bar{X}}A_2$, (2) $\underparen{C_1\bar{\bar{Z}}C_2}$ is a subset of E_{12}, (3) I_1 and I_2 are either both within or both without $A_1B_1C_1\bar{\bar{Z}}C_2B_2A_2\bar{\bar{X}}A_1$.

If no point of A_1K_2 (on $A_1X_2A_2$) is on $C_1\bar{\bar{Z}}C_2$, then by Theorem E, I_1 and I_2 are either both within or both without $A_1B_1C_1\bar{\bar{Z}}C_2B_2A_2X_2A_1$. Suppose now that there is a point of AK_2 (on $A_1X_2A_2$) on $C_1\bar{\bar{Z}}C_2$. Let S_1 be the first such point. It is evident that K_1 is the last point of the arc $A_1\bar{\bar{X}}A_2$ on the arc A_1S_1 of $A_1X_2A_2$, while K_2 will be the first point of the arc S_1A_2 of $A_1X_2A_2$ on $A_1\bar{\bar{X}}A_2$. Proceeding now with S_1 as we did with L_1 in Theorem F, it is evident that we will get a closed curve $A_1B_1C_1Z'C_2B_2A_2X'A_1$ having all the properties of the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$ of Theorem F and having the additional property that $n = 1$. Hence by the proof carried on in the first part of this theorem, there exists an arc $C_1Z_2C_2$ failing to meet $A_1X_2A_2$ and such that (1) $\underparen{C_1Z_2C_2}$ is a subset of E_{12}, (2) I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$. Thus, if we assume our theorem is true when $n = k - 1$, it is also true when $n = k$. But it is true when $n = 1$. Hence our induction is complete.

Theorem H: *Suppose* (1) A_i, B_i *and* C_i *are distinct points of the closed curve* J_i $(i = 1, 2)$, (2) $J_1 + I_1$ *is a subset of* E_2, (3) $A_1X_1A_2$, $A_1X_2A_2$, $C_1Z_1C_2$ *and* $C_1Z_2C_2$ *are arcs which lie except for their endpoints in* E_{12}, (4) $A_1X_iA_2$ *and* $C_1Z_iC_2$ $(i = 1, 2)$ *have no points in common,* (5) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. *Under these conditions,* I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Proof: By Theorem G there exists at least one arc C_1EC_2 such that (1) C_1EC_2 fails to meet $A_1X_2A_2$, (2) $\underparen{C_1EC_2}$ is a subset of E_{12}, (3) I_1 and I_2 are either both within or both without $A_1B_1C_1EC_2B_2A_2X_2A_1$. Hence, by Theorem E, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Definition: If A_i, B_i and C_i are distinct points of J_i $(i = 1, 2)$, where J_1 and J_2 are closed curves such that $J_1 + I_1$ is a subset of E_2, then we shall say that $A_1B_1C_1$ can be *simply joined* to $A_2B_2C_2$ if there exist arcs

A_1XA_2, B_1YB_2 and C_1ZC_2, which are such that no two have any point in common and, which lie, except for their endpoints, in E_{12}.

Theorem I: *Suppose* (1) J_1, J_2 *and* J_3 *are closed curves such that* $J_i + I_i$ *is a subset of* $E_{i+1,\, i+2}$,[*] (2) A_i, B_i *and* C_i $(i = 1, 2, 3)$ *are distinct points of* J_i, (3) $A_1B_1C_1$ *can be simply joined to* $A_3B_3C_3$ *and* $A_2B_2C_2$ *can be simply joined to* $A_3B_3C_3$. *Then* $A_1B_1C_1$ *and* $A_2B_2C_2$ *cannot be simply joined.*

Proof: Draw arcs $A_1X_{13}A_3$ and $C_1Z_{13}C_3$, having no points in common and such that $\underset{\smile}{A_1X_{13}A_3} + \underset{\smile}{C_1Z_{13}C_3}$ is a subset of E_{123}, the common exterior of J_1, J_2 and J_3 and such that J_2 is without the closed curve $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$.[†] By Theorems B and H, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Let D_i denote a point of J_i such that A_i and C_i separate B_i and D_i on J_i. There are two cases possible.

Case I. The sets I_1 and I_3 are both without $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Then it may easily be proved that the interior of $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$ $= \underset{\smile}{A_1B_1C_1} + \underset{\smile}{A_3B_3C_3} + I_1 + I_3 +$ the interior of $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Draw arcs $A_2X_{23}A_3$ and $C_2Z_{23}C_3$, having no points in common and such that $\underset{\smile}{A_2X_{23}A_3} + \underset{\smile}{C_2Z_{23}C_3}$ is a subset of the common exterior of J_2 and $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. There are two possibilities.

(*a*) The point D_1 is without $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. Then the interior of $A_1D_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$ is $\underset{\smile}{A_3D_3C_3}$ + the interior of $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$ + the interior of $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. As $\underset{\smile}{A_3B_3C_3}$ is without the closed curve $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$ while B_2 is not within $A_3B_3C_3D_3A_3$, it follows that I_3 is within $A_3B_3C_3Z_{23}C_2B_2A_2X_{23}A_3$.[‡] Hence, by Theorems B and H, I_2 must be within $A_3B_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. As $\underset{\smile}{A_1B_1C_1}$ is within $A_1D_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, then I_1 is without $A_1B_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, which we know does contain I_2. Hence, by Theorems B and H, we may conclude that $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.

(*b*) The point D_1 is within $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$, whose interior then is $\underset{\smile}{A_3X_{13}A_1D_1C_1Z_{13}C_3}$ + the interior of $A_2B_2C_2Z_{23}C_3Z_{13}C_1D_1A_1X_{13}A_3X_{23}A_2$ + the interior of $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. As $\underset{\smile}{A_3B_3C_3}$ is within $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$, I_3 is without $A_3B_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. Hence, by Theorems B and H, I_2 is also without $A_3B_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. As $\underset{\smile}{A_1B_1C_1}$ is without $A_1D_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, while B_2 is without $A_1B_1C_1D_1A_1$, I_1 is within $A_1B_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, which we know fails to contain I_2. Hence, by Theorems B and H, $A_1B_1C_1$ and $A_2B_2C_2$ are not able to be simply joined.

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[*] It is understood throughout this and the following theorem, that subscripts are reduced modulo 3.

[†] That this drawing of arcs is possible, may easily be proved on the basis of Σ_3.

[‡] Cf. R. L. Moore, loc. cit., Theorem 27, pp. 144–5.

Case II. I_1 and I_3 are both within $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Then I_1 and I_3 are both without $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. As $A_1B_1C_1$ and $A_3B_3C_3$ are able to be simply joined, $A_1D_1C_1$ and $A_3D_3C_3$ can also be simply joined, by Theorem C. In like manner, $A_2D_2C_2$ and $A_3D_3C_3$ can be simply joined. By Case I, $A_1D_1C_1$ and $A_2D_2C_2$ cannot be simply joined. Hence, by Theorem C, $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.

Theorem J: *Suppose* J_1, J_2 *and* J_3 *are closed curves such that* $J_i + I_i$ $(i = 1, 2, 3)$ *is a subset of* $E_{i+1,\ i+2}$, (2) A_i *and* C_i *separate* B_i *and* D_i *on* J_i, (3) $A_3B_3C_3$ *cannot be simply joined to either* $A_1B_1C_1$ *or* $A_2B_2C_2$. *Then* $A_1B_1C_1$ *and* $A_2B_2C_2$ *cannot be simply joined.*

Proof: Join A_1 to A_3 and C_1 to C_3 by arcs, $A_1X_{13}A_3$ and $C_1Z_{13}C_3$ which have no point in common and lie except for their endpoints in E_{13}. As $A_1B_1C_1$ and $A_3B_3C_3$ cannot be simply joined, it follows, by Theorems B and H, that either I_1 is within and I_3 is without or I_1 is without and I_3 is within $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. It follows that I_1 and I_3 are either both without or both within $A_1B_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. Hence $A_1B_1C_1$ and $A_3D_3C_3$ can be simply joined. In like manner $A_2B_2C_2$ and $A_3D_3C_3$ can be simply joined. It follows, by Theorem I, that $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.

Theorem K: *Suppose* J_1 *and* J_2 *are closed curves such that* $J_1 + I_1$ *is a subset of* E_2, (2) A_i *and* C_i *separate* B_i *and* D_i *on* J_i, (3) $A_1B_1C_1$ *and* $A_2B_2C_2$ *can be simply joined. Then* $A_1B_1C_1$ *and* $C_2B_2A_2$ *cannot be simply joined. Conversely if* $A_1B_1C_1$ *and* $C_2B_2A_2$ *cannot be simply joined, then* $A_1B_1C_1$ *and* $A_2B_2C_2$ *can be simply joined.*

Proof: Join A_1 to A_2 and C_1 to C_2 by arcs, $A_1X_2A_2$ and $C_1Z_2C_2$, which have no point in common and lie except for their endpoints in E_{12}. As $A_1B_1C_1$ and $A_2B_2C_2$ can be simply joined, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$. Let us suppose they are both without. Then it may be proved that the interior of $A_1D_1C_1Z_2C_2D_2A_2X_2A_1 = \underparen{A_1B_1C_1} + \underparen{A_2B_2C_2} + I_1 + I_2 +$ the interior of $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$. Let $A_1M_1C_2$ denote an arc such that $\underparen{A_1M_1C_2}$ is within $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$ while $C_1K_1A_2$ is an arc such that $\underparen{C_1K_1A_2}$ is without $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. There are two possibilities.

(a) X_2 is within $A_2D_2C_2Z_2C_1K_1A_2$, whose interior then is $\underparen{A_2X_2A_1D_1C_1}$ + interior of $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$ + the interior of $A_2X_2A_1D_1C_1K_1A_2$. As $\underparen{C_1B_1A_1M_1C_2}$ is within $A_2D_2C_2Z_2C_1K_1A_2$, the interior of $A_2D_2C_2Z_2C_1K_1A_2 = \underparen{C_1B_1A_1M_1C_2}$ + interior of $C_1B_1A_1M_1C_2Z_2C_1$ + interior of A_1-

$B_1C_1K_1A_2D_2C_2M_1A_1$. As the interior of $C_1B_1A_1M_1C_2Z_2C_1$ is a subset of the interior of $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$,* it follows that I_1 and I_2 together with $\underparen{A_2B_2C_2}$ are within $A_1B_1C_1K_1A_2D_2C_2M_1A_1$. As $\underparen{A_2B_2C_2}$ is within $A_1B_1C_1K_1A_2D_2C_2M_1A_1$, it follows that I_2, the interior of $A_2B_2C_2D_2A_2$ is without $A_1B_1C_1K_1A_2B_2C_2M_1A_1$.† Hence I_1 is within and I_2 is without $A_1B_1C_1K_1A_2B_2C_2M_1A_1$.

(*b*) The point X_2 is without $A_2D_2C_2Z_2C_1K_1A_2$. Then the interior of $A_1D_1C_1K_1A_2X_2A_1 = \underbracket{A_2D_2C_2Z_2C_1}$ + the interior of $A_2D_2C_2Z_2C_1K_1A_2$ + the interior of $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. As $\underbracket{A_1M_1C_2B_2A_2}$ is within $A_1D_1C_1K_1A_2X_2A_1$, the interior of $A_1D_1C_1K_1A_2X_2A_1 = \underbracket{A_1M_1C_2B_2A_2}$ + the interior of $A_1M_1C_2B_2A_2X_2A_1$ + the interior of $A_2B_2C_2M_1A_1D_1C_1K_1A_2$. As I_1, I_2 and $\underparen{A_1B_1C_1}$ are in $A_1D_1C_1K_1A_2X_2A_1$ and not in $A_1M_1C_2B_2A_2X_2A_1$, which lies entirely within $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$, it follows that I_1, I_2 and $\underparen{A_1B_1C_1}$ are within $A_2B_2C_2M_1A_1D_1C_1K_1A_2$. As $\underparen{A_1B_1C_1}$ is within $A_2B_2C_2M_1A_1D_1C_1K_1A_2$, it follows that I_1, the interior of $A_1B_1C_1D_1A_1$ is without $A_2B_2C_2M_1A_1B_1C_1K_1A_2$, which does contain I_2.

Thus, if I_1 and I_2 are both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$, it follows that $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply joined. If I_1 and I_2 are both within $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$, I_1 and I_2 are both without $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. Hence $A_1D_1C_1$ and $C_2D_2A_2$ cannot be simply joined. Hence it follows, by Theorem *C*, that $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply joined.

A similar argument will show that if $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply connected, then $A_1B_1C_1$ and $A_2B_2C_2$ can be simply connected.

3. Properties of Sense.

Theorem 1: *If for one choice of the closed curve J_3 in E_{12} and three distinct points $A_3B_3C_3$ thereon, $A_1B_1C_1$ on J_1 and $A_2B_2C_2$ on J_2 are either both able or both not able to be simply joined to $A_3B_3C_3$, then for any other closed J_4 in E_{12} and three distinct points $A_4B_4C_4$ thereon $A_1B_1C_1$ and $A_2B_2C_2$ are either both able or both not able to be simply joined to $A_4B_4C_4$.*

Proof: Case I. Suppose $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined. Then $A_2B_2C_2$ and $A_3B_3C_3$ can be simply joined. Choose in the common exterior of J_1, J_2, J_3 and J_4 a closed curve J_5 and three points A_5, B_5 and C_5 thereon such that $A_3B_3C_3$ and $A_5B_5C_5$ can be simply joined. By Theorem *I*, $A_5B_5C_5$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$. There are two possibilities

(*a*) $A_4B_4C_4$ and $A_5B_5C_5$ can be simply joined. Then, by Theorem *K*,

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[*] Cf. R. L. Moore, loc. cit., Theorem 21, p. 140.

[†] Cf. R. L. Moore, loc. cit., Theorem 24, p. 141.

$C_4B_4A_4$ and $A_5B_5C_5$ cannot be simply joined. By Theorem J, $C_4B_4A_4$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$. By Theorem K, $A_4B_4C_4$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

(b) $A_4B_4C_4$ and $A_5B_5C_5$ cannot be simply joined. Then, by Theorem J, $A_4B_4C_4$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$.

Case II. Suppose $A_1B_1C_1$ and $A_3B_3C_3$ cannot be simply joined. Then $A_2B_2C_2$ and $A_3B_3C_3$ cannot be simply joined. Then, by Theorem K, $A_3B_3C_3$ can be simply joined to both $C_1B_1A_1$ and $C_2B_2A_2$. Then by an argument, similar to that used in Case I, we may show that either (a) $A_4B_4C_4$ can be simply joined to both $C_1B_1A_1$ and $C_2B_2A_2$ or (b) $A_4B_4C_4$ cannot be simply joined to either $C_1B_1A_1$ or $C_2B_2A_2$. By Theorem K, either $A_4B_4C_4$ (a) can be simply joined to neither $A_1B_1C_1$ nor $A_2B_2C_2$ or (b) can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

Theorem 2: *If the senses $A_1B_1C_1$ on J_1 and $A_2B_2C_2$ on J_2 are the same with respect to their common exterior E_{12} and the senses $A_2B_2C_2$ on J_2 and $A_3B_3C_3$ on J_3 are the same with respect to their common exterior E_{23}, then the senses $A_1B_1C_1$ on J_1 and $A_3B_3C_3$ on J_3 are the same with respect to their common exterior E_{13}.*

Proof: Choose a closed curve J_4 in the common exterior of J_1, J_2 and J_3 and three distinct points A_4, B_4 and C_4 on J_4 such that $A_1B_1C_1$ and $A_4B_4C_4$ can be simply joined. As the sense $A_1B_1C_1$ on J_1 and the sense $A_2B_2C_2$ on J_2 are the same with respect to E_{12}, it follows, by definition and Theorem 1, that $A_2B_2C_2$ and $A_4B_4C_4$ can be simply joined. As the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3 with respect to E_{23}, it follows, by definition and Theorem 1, that $A_3B_3C_3$ and $A_4B_4C_4$ can be simply joined. Hence, by definition, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_3B_3C_3$ on J_3 with respect to E_{13}.

Theorem 3: *If the sense $A_1B_1C_1$ on J_1 is opposite to the sense $A_2B_2C_2$ on J_2 with respect to E_{12}, and the sense $A_2B_2C_2$ on J_2 is opposite to the sense $A_3B_3C_3$ on J_3 with respect to E_{23}, then the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_3B_3C_3$ with respect to E_{13}.*

Proof: Choose in the common exterior of J_1, J_2 and J_3 a closed curve J_4 and three distinct points A_4, B_4 and C_4 on J_4 such that $A_4B_4C_4$ and $A_1B_1C_1$ can be simply joined. As the sense $A_1B_1C_1$ on J_1 is opposite to the sense $A_2B_2C_2$ on J_2 with respect to E_{12}, it follows that $A_2B_2C_2$ and $A_4B_4C_4$ cannot be simply joined. As the senses $A_2B_2C_2$ on J_2 and $A_3B_3C_3$ on J_3 are opposite with respect to E_{23}, it follows that $A_3B_3C_3$ and $A_4B_4C_4$ can be simply joined. As $A_4B_4C_4$ can be simply joined to $A_3B_3C_3$ and to $A_1B_1C_1$, it follows that the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_3B_3C_3$ on J_3 with respect to E_{13}.

Theorem 4: *The sense $A_1B_1C_1$ on J_1 is opposite to the sense $C_1B_1A_1$ on J_1 with respect to E_1.*

The truth of Theorem 4 follows at once from our definition and Theorems 1 and K.

Theorem 5: *If the closed curve* J_1 *is a subset of the interior of the closed curve* J_2*, then, in order that the sense* $A_1B_1C_1$ *on* J_1 *and the sense* $A_2B_2C_2$ *on* J_2 *be the same with respect to* E_{12}*, it is both necessary and sufficient that there exist arcs* A_1XA_2*,* B_1YB_2 *and* C_1ZC_2*, no two having any points in common and lying except for their endpoints entirely between** J_1 *and* J_2.

Proof: *The condition is sufficient.* Let J_3 denote any closed curve in E_2, the exterior of J_2. Choose on J_3 points A_3, B_3 and C_3 such that $A_3B_3C_3$ and $A_2B_2C_2$ can be simply joined. Hence there exist arcs $A_2X_2A_3$, $B_2Y_2B_3$ and $C_2Z_2C_3$ such that (1) these arcs lie except for their endpoints in E_{23}, (2) no two have any points in common, (3) I_2 and I_3 are both without $A_2B_2C_2Z_2C_3B_3A_3X_2A_2$. By hypothesis, there exist arcs B_1YB_2, A_1XA_2 and C_1ZC_2, no two of which have any point in common and lie except for their endpoints within J_2 but without J_1. Hence $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined. Hence the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12}.

The condition is necessary. Suppose the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12}. Consider the closed curve J_3 and the arcs $A_2X_2A_3$, $B_2Y_2B_3$ and $C_2Z_2C_3$ described above. Draw arcs $A_1X_1A_2$ and $C_1Z_1C_2$ which have no point in common and lie except for their endpoints in between J_1 and J_2. As the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2, it follows that, as $A_2B_2C_2$ and $A_3B_3C_3$ can be simply joined, $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined. As $\underbracket{A_2X_1A_1B_1C_1Z_1C_1}$ is without $A_3B_3C_3Z_2C_2B_2A_2X_2A_3$, and B_3 is not within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$, then $\underparen{A_2B_2C_2}$ is within $A_1B_1C_1Z_1$-$C_2Z_2C_3B_3A_3X_2A_2X_1A_1$, whose interior is $\underparen{A_2B_2C_2}$ + the interior of $A_2B_2C_2$-$Z_1C_1B_1A_1X_1A_2$ + the interior of $A_2B_2C_2Z_2C_3B_3A_3X_2A_2$. As $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined while I_3 is without $A_1B_1C_1Z_1C_2Z_2C_3B_3A_3$-$X_2A_2X_1A_1$, it follows by Theorems B and H, that I_1 is without $A_1B_1C_1$-$Z_1C_2Z_2C_3B_3A_3X_2A_2X_1A_1$ and hence without $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$. But all points within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$ are within J_2. Hence all points within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$ are between J_1 and J_2. But there exists an arc $B_1Y_1B_2$ such that $\underparen{B_1Y_1B_2}$ is within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$. Hence, if the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12} and J_1 is in the interior of J_2, there exist arcs $A_1X_1A_2$, $B_1Y_1B_2$ and $C_1Z_1C_2$, no two of which have any point in common and which lie except for their endpoints between J_1 and J_2.

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* A point set M is said to lie *between* two closed curves if one of the curves lies within the other and M lies without the first and within the second one.